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Abstract. We use the collective-field method to discuss the weak- and strong-coupling phases of the one-plaquette U(N)-invariant model. In both phases we obtain a unified description in terms of collective fields and their correlations. The ground-state energy including the next-to-leading-order term is finite owing to explicit cancellation of divergences except at the critical point of the phase transition.

1. Introduction

The large-N approximation to quantum field theories can be efficiently extracted using the collective-field method. Gross and Witten [1] applied the $1/N$-expansion method to the two-dimensional U(N) gauge-invariant lattice field theory. They discovered a third-order phase transition.

In this paper we apply the collective-field method to the one-plaquette U(N) gauge-invariant model including divergent terms in the Hamiltonian which are inherent to the collective-field formulation [2, 3]. We show that the divergences in the collective Hamiltonian are cancelled by the divergent contributions of zero-point collective fluctuations from the next-to-leading order in the $1/N$ expansion. The remaining term represents the finite correction to the ground-state energy. In terms of collective fields we obtain a unified description of both phases. It is only the semiclassical solution ϕ_0 and the correlation that turn out to be different functions in different regimes of the coupling constant. In the vicinity of the critical value of the coupling constant the energy becomes divergent.

We show that in the strong-coupling sector, at least for the chosen cases, the collective-field method applied to lattice gauge theories seems to be superior to the summation method of fermionic orbitals used by Brezin, Itzykson, Parisi and Zuber (BIPZ) [4].

2. The collective-field Hamiltonian

The one-plaquette Kogut–Susskind Hamiltonian [5] with U(N) gauge symmetry in 2+1 dimensions is given by

$$H=\frac{g^2}{2a}\left[\sum_{i=1,\alpha}^{4} E^{\alpha}(i)E^{\alpha}(i)+\frac{2}{g^4}S(\mathrm{U}(1)\mathrm{U}(2)\mathrm{U}(3)\mathrm{U}(4))\right] \qquad \alpha=0,1,\ldots,N^2-1 \quad (1)$$

where g is the coupling constant and a is the lattice spacing. The unitary matrices $U(i)$ are the basic degrees of freedom and the electric field $E^{\alpha}(i)$ is expressed by the conjugate variables in the vertex i of the plaquette. The lattice action S is a real class function defined on the group $U(N)$. For example, the action S can have the following forms:

$$S=\begin{cases}-[\operatorname{Tr} U(1)U(2)U(3)U(4)+\text{HC}] & (2a)\\ \operatorname{Tr}\chi^2 & (2b)\\ \operatorname{Tr}\tanh^2(\chi/2) & (2c)\end{cases}$$

where

$$e^{i\chi}\equiv U(1)U(2)U(3)U(4). \tag{3}$$

The corresponding models are known as the Wilson [5], Manton [6] and Jurkiewicz-Zalewski [7] actions. We shall investigate the ground-state energy in the large-N limit. A suitable method for extracting leading terms is the collective-field method. We shall therefore formulate the Hamiltonian in terms of collective fields, as was done in [2] for the leading term in the $1/N$ expansion. In addition, we shall include the next-to-leading term in the collective-field Hamiltonian.

For the collective field we use commuting gauge-invariant 'loop-space' operators

$$w_n=\operatorname{Tr}\{[U(1)U(2)U(3)U(4)]^n\} \tag{4}$$

where n is an integer.

With use of the standard procedure of Hermiticization, the kinetic part of the Hamiltonian is

$$\begin{aligned}H_{\text{kin}}&=\sum_{\alpha,i}E^{\alpha}(i)E^{\alpha}(i)\\ &=\frac{2g^2}{a}\left(-\tfrac{1}{2}\sum_{nn'}\frac{\partial}{\partial w_n}\Omega(n,n')\frac{\partial}{\partial w_{n'}}+\tfrac{1}{8}\sum_{nn'}\omega(n)\Omega^{-1}(n,n')\omega(n')\right)\\ &\quad-\tfrac{1}{4}\sum_n\frac{\partial}{\partial w_n}\left(nNW_n+n\sum_{n'}\operatorname{sgn} n'\,W_nW_{n-n'}\right)\end{aligned} \tag{5}$$

where

$$\Omega(n,n')=-nn'W_{n+n'} \tag{6a}$$

and

$$\omega(n)=n\left(NW_n+\sum_{n'=\operatorname{sgn} n}^{n-\operatorname{sgn} n}W_{n'}W_{n-n'}\right). \tag{6b}$$

The last term in (5) is divergent, and we shall show how to cope with such divergences.

In the large-N it is convenient to work with the continuous version of the collective field (4) by using a fourier transform:

$$\phi(\sigma)=\sum_{-\infty}^{\infty}\frac{1}{2\pi}e^{in\sigma}W_n \qquad \sigma\varepsilon(-\pi,\pi). \tag{7}$$

Then we obtain

$$\Omega(\sigma,\sigma')=\partial_\sigma\partial_{\sigma'}(\delta(\sigma-\sigma')\phi(\sigma)) \tag{8a}$$

$$\omega(\sigma)=-\partial_\sigma\left(\phi(\sigma)\fint d\sigma'\cot\frac{\sigma-\sigma'}{2}\phi(\sigma')\right) \tag{8b}$$

and the kinetic part of the Hamiltonian (5)

$$H_{\rm kin}=\frac{2g^2}{a}\fint \mathrm{d}\sigma\left(\partial_0\pi\phi(\sigma)\partial_\sigma\pi+\tfrac{1}{8}\phi(\sigma)\left(\fint \mathrm{d}\sigma'\,\phi(\sigma')\cot\frac{\sigma-\sigma'}{2}\right)^2\right)$$
$$-\frac{g^2}{2a}\fint \mathrm{d}\sigma\,\phi(\sigma)\partial_\sigma\left(\cot\frac{\sigma-\sigma'}{2}\right)_{\sigma=\sigma'} \tag{9}$$

where

$$\pi(\sigma)=-\mathrm{i}\frac{\delta}{\delta\phi(\sigma)}.$$

The collective field $\phi(\sigma)$ obeys the normalization condition

$$\int \mathrm{d}\sigma\phi(\sigma)=\mathrm{Tr}\,\mathbb{1}=N. \tag{10}$$

Owing to the functional identity

$$\int \phi(\sigma)\left(\fint \mathrm{d}\sigma'\cot\frac{\sigma'-\sigma}{2}\phi(\sigma')\right)^2\mathrm{d}\sigma=\frac{4\pi^2}{3}\int \phi^3(\sigma)\,\mathrm{d}\sigma-\tfrac{1}{3}\left(\int \phi(\sigma)\,\mathrm{d}\sigma\right)^3 \tag{11}$$

(this can be proved using arguments similar to those in [8]), we can write $H_{\rm kin}$ as the local functional:

$$H_{\rm kin}=\frac{2g^2}{a}\left(\tfrac{1}{2}\int \mathrm{d}\sigma\,\partial_\sigma\pi\phi(\sigma)\partial_\sigma\pi+\frac{\pi^2}{6}\int \phi^3(\sigma)\,\mathrm{d}\sigma-\tfrac{1}{24}\left(\int \phi(\sigma)\,\mathrm{d}\sigma\right)^3\right)$$
$$-\frac{g^2}{2a}\fint \mathrm{d}\sigma\,\phi(\sigma)\partial_\sigma\left(\cot\frac{\sigma-\sigma'}{2}\right)_{\sigma'=\sigma}. \tag{12}$$

The lattice action S can be written in the general form

$$S=\int_{-\pi}^{\pi}\phi(s)S(\sigma)\,\mathrm{d}\sigma \tag{13}$$

where

$$S(\sigma)=\begin{cases}-2\cos\sigma & (14a)\\ \sigma^2 & (14b)\\ 2\tan^2\dfrac{\sigma}{2} & (14c)\end{cases}$$

for the Wilson, Manton and Jurkiewicz–Zalewski actions, respectively.

The complete Hamiltonian is

$$H=\frac{2g^2}{a}\left(\frac{1}{2N}\int \mathrm{d}\sigma\,\partial_\sigma\pi\phi(\sigma)\partial_\sigma\pi+\frac{\pi^2N^3}{6}\int \phi^3(\sigma)\,\mathrm{d}\sigma-\frac{N^3}{24}\left(\int \phi(\sigma)\,\mathrm{d}\sigma\right)^3\right)$$
$$-\frac{g^2N}{2a}\int \mathrm{d}\sigma\,\phi(\sigma)\partial_\sigma\left(\cot\frac{\sigma-\sigma'}{2}\right)_{\sigma=\sigma'}+\frac{N}{g^2a}\int \phi(\sigma)S(\sigma)\,\mathrm{d}\sigma. \tag{15}$$

We shall perform the large-N expansion with the standard condition

$$g^2N=\lambda=\text{constant}.$$

From now on we shall extract N from the collective field, so that the normalization condition (10) is

$$\int_{-\pi}^{\pi} \phi(\sigma)\,\mathrm{d}\sigma = 1. \tag{16}$$

With this in mind, we can write the Hamiltonian (15) so that its explicit N dependence is

$$H = N^2\left(\frac{\pi^2}{3a}\lambda \int \phi^3(\sigma)\,\mathrm{d}\sigma - \frac{\lambda}{12a}\left(\int \phi(\sigma)\,\mathrm{d}\sigma\right)^3 + \frac{1}{\lambda a}\int \phi(\sigma)S(\sigma)\right)$$
$$-\frac{\lambda}{2a}\int \mathrm{d}\sigma\,\phi(\sigma)\partial_\sigma\left(\cot\frac{\sigma-\sigma'}{2}\right)_{\sigma=\sigma'} + \frac{\lambda}{aN^2}\int \partial_\sigma\pi\phi(\sigma)\partial_\sigma\pi\,\mathrm{d}\sigma. \tag{17}$$

From the structure of the Hamiltonian we assume a Gaussian ansatz for the vacuum wavefunctional [9]:

$$\Phi[\phi] = \exp\left(-\frac{N^2}{4}\int \mathrm{d}\sigma\,\mathrm{d}\sigma'(\phi(s)-\phi_0(\sigma))G^{-1}(\sigma,\sigma')(\phi(\sigma')-\phi_0(\sigma'))\right). \tag{18}$$

In our variational approach, two important variables are the ground-state mean value of the field

$$\langle\Phi|\phi(\sigma)|\Phi\rangle = \phi_0(\sigma) \tag{19}$$

and the correlation $G(\sigma, \sigma')$

$$\langle\Phi|\phi(\sigma)\phi(\sigma')|\Phi\rangle = \phi_0(\sigma)\phi_0(\sigma') + G(\sigma,\sigma') \tag{20}$$

where G and G^{-1} are related by

$$\int G(\sigma,\sigma')G^{-1}(\sigma',\sigma'')\,\mathrm{d}\sigma' = \delta(\sigma-\sigma''). \tag{21}$$

The ground-state energy given in terms of $\phi_0(\sigma)$ and $G(\sigma,\sigma')$ is

$$E = \langle\Phi|H|\Phi\rangle = \frac{\pi^2\lambda N^2}{3a}\int \phi_0^3(\sigma)\,\mathrm{d}\phi + \frac{N^2}{\lambda a}\int \phi_0(\sigma)S(\sigma)\,\mathrm{d}\sigma - \frac{\lambda N^2}{12a}$$
$$+\frac{\pi^2\lambda N^2}{a}\int \phi_0(\sigma)G(\sigma,\sigma)\,\mathrm{d}\sigma - \frac{\lambda}{2a}\int \mathrm{d}\sigma\,\phi_0(\sigma)\partial_\sigma\left(\cot\frac{\sigma-\sigma'}{2}\right)_{\sigma=\sigma'}$$
$$+\frac{\lambda N^2}{4a}\int \mathrm{d}\sigma\,\phi_0(\sigma)\partial_\sigma\partial_{\sigma'}G^{-1}(\sigma,\sigma')_{\sigma'=\sigma}. \tag{22}$$

In order to find the extreme of the energy functional, we shall find functional variations with respect to $\phi_0(\sigma)$ and $G(\sigma,\sigma')$. We obtain two coupled nonlinear equations:

$$\frac{\pi^2\lambda N^2}{a}\phi_0^2(\sigma) + \frac{N^2}{\lambda a}S(\sigma) + \frac{\pi^2\lambda N^2}{a}G(\sigma,\sigma) - \frac{\lambda}{2a}\partial_\sigma\cot\frac{\sigma-\sigma'}{2}\Big|_{\sigma'=\sigma}$$
$$+\frac{\lambda N^2}{4a}\partial_\sigma\partial_{\sigma'}G^{-1}(\sigma,\sigma')_{\sigma'=\sigma} = \mu_1 \tag{23a}$$

$$\frac{\pi^2\lambda}{a}\phi_0(\sigma')\delta(\sigma'-\sigma'') - \frac{\lambda}{4a}\int \mathrm{d}\sigma\,\phi_0(\sigma)\partial_\sigma G^{-1}(\sigma,\sigma')\partial_\sigma G^{-1}(\sigma,\sigma'') = \mu_2 \tag{23b}$$

where μ_1 and μ_2 are Lagrangian multipliers owing to the 'normalization' conditions

$$\int \phi_0(\sigma)\,\mathrm{d}\sigma = 1 \tag{24}$$

$$\int \mathrm{d}\sigma\, G(\sigma,\sigma') = \int \mathrm{d}\sigma'\, G(\sigma,\sigma') = 0 \tag{25}$$

and the energy functional (22) can be written as

$$E = \frac{\pi^2\lambda N^2}{3a}\int \phi_0^3(\sigma)\,\mathrm{d}\sigma + \frac{N^2}{\lambda a}\int \phi_0(\sigma)S(\sigma)\,\mathrm{d}\sigma - \frac{\lambda N^2}{12a}$$
$$+\frac{\lambda}{2a}\int \mathrm{d}\sigma\,\phi_0(\sigma)\left[\partial_\sigma\partial_{\sigma'}G^{-1}(\sigma,\sigma')_{\sigma'=\sigma} - \partial_\sigma\left(\cot\frac{\sigma-\sigma'}{2}\right)_{\sigma=\sigma'}\right]. \tag{26}$$

The functional identity [8] (taken in the sense of principal part)

$$\cot\frac{\sigma-\sigma'}{2}\cot\frac{\sigma-\sigma''}{2} + \cot\frac{\sigma'-\sigma}{2}\cot\frac{\sigma'-\sigma''}{2} + \cot\frac{\sigma''-\sigma}{2}\cot\frac{\sigma''-\sigma'}{2}$$
$$= 4\pi^2\delta(\sigma-\sigma')\delta(\sigma-\sigma'') - 1 \tag{27}$$

can be extended by mapping $\sigma \to f(\sigma)$, where $f(\sigma)$ is a continuous real function satisfying $-\pi \leqslant f(\sigma) \leqslant \pi$. Using the identity (27) we obtain

$$⨍ \mathrm{d}\sigma\,\phi_0(\sigma)(f'(\sigma))^2\cot\frac{f(\sigma)-f(\sigma')}{2}\cot\frac{f(\sigma)-f(\sigma'')}{2}$$
$$= 4\pi^2\phi_0(\sigma')\delta(\sigma'-\sigma'') - \int \phi_0(\sigma)(f'(\sigma))^2\,\mathrm{d}\sigma - \cot\frac{f(\sigma')-f(\sigma'')}{2}$$
$$\times\left[⨍ \phi_0(\sigma)(f'(\sigma))^2\,\mathrm{d}\sigma\,\cot\frac{f(\sigma')-f(\sigma)}{2} - ⨍ \phi_0(\sigma)\right.$$
$$\left.\times(f'(\sigma))^2\,\mathrm{d}\sigma\,\cot\frac{f(\sigma'')-f(\sigma)}{2}\right].$$

If the term in the brackets vanishes, we find the solution $G^{-1}(\sigma,\sigma')$ in (23*b*),

$$G^{-1}(\sigma,\sigma') = -2\ln\left|\sin\frac{f(\sigma)-f(\sigma')}{2}\right| \tag{28}$$

and the condition on $f(\sigma)$ arsises from

$$⨍_{-\pi}^{\pi} \phi_0(\sigma)(f'(\sigma))^2\,\mathrm{d}\sigma\,\cot\frac{f(\sigma')-f(\sigma)}{2} = \text{constant}. \tag{29}$$

From this condition we may conclude that $\phi_0(\sigma)f'(\sigma) = \text{constant}$, owing to the property of the cotangent kernel [10]. Therefore, choosing $f(\pm\pi) = \pm\pi$, we obtain

$$f(\sigma) = -\pi + \frac{2\pi t}{T} \tag{30}$$

where

$$t(\sigma)=\int_{-\pi}^{\sigma}\frac{\mathrm{d}\sigma'}{(\sigma')}\qquad\text{and}\qquad T=t(\pi) \tag{31}$$

and

$$G^{-1}(\sigma,\sigma')=-2\ln\left|\sin\frac{\pi}{T}(t(\sigma)-t(\sigma'))\right|. \tag{32}$$

Inserting (32) into the expression for the ground-state energy (22), we obtain

$$E=\frac{\pi^2\lambda N^2}{3a}\int\phi_0^2(\sigma)\,\mathrm{d}\sigma+\frac{N^2}{\lambda a}\int\phi_0(\sigma)S(\sigma)\,\mathrm{d}\sigma-\frac{\lambda N^2}{12a}$$
$$-\frac{\lambda}{a}\int\mathrm{d}\sigma\,\phi_0(\sigma)\partial_\sigma\partial_{\sigma'}\ln\left|\frac{\sin(\pi/T)(t(\sigma)-t(\sigma'))}{\sin(\sigma-\sigma')/2}\right|_{\sigma=\sigma'}. \tag{33}$$

The last term is a correction of order $1/N^2$ with respect to the leading term:

$$E_{\text{corr}}=\frac{\lambda}{4a}\partial_\sigma\phi_0(\sigma)\,\Big|_{\sigma=-\pi}^{\sigma=\pi}-\frac{\lambda}{24a}\int\frac{\partial_\sigma^2\phi_0^2}{\phi_0}\,\mathrm{d}\sigma-\frac{\pi^2\lambda}{3Ta}+\frac{\lambda}{12a}. \tag{34}$$

Inserting the expression for $G^{-1}(\sigma,\sigma')$ into (23a), which determines $\phi_0(\sigma)$, we obtain

$$\frac{\pi^2\lambda N^2}{a}\phi_0^2(\sigma)+\frac{N^2}{\lambda a}S(\sigma)-\frac{\lambda}{a}\partial_\sigma\partial_{\sigma'}\ln\left|\frac{\sin(\pi/T)(t(\sigma)-t(\sigma'))}{\sin(\sigma-\sigma')/2}\right|_{\sigma'=\sigma}=\mu_1. \tag{35}$$

Generally, this equation can be exactly solved only in a few cases, as we shall discuss later. Using the $1/N$ expansion we can solve (35) approximately. The term with the derivatives is suppressed by $1/N^2$ with respect to the leading term, so that the approximate solution in the leading order is

$$\tilde{\phi}_0(\sigma)=\frac{1}{\pi\lambda}\sqrt{\mu-S(\sigma)}. \tag{36}$$

This is sufficient to obtain the ground-state energy and its correction. In fact, the collective field $\tilde{\phi}_0(\sigma)$ is the extremal point of the leading term in the energy functional (26).

From (23b) we obtain the condition

$$\phi_0(\sigma)\partial_\sigma G^{-1}(\sigma,\sigma')\,\Big|_{\sigma=-\pi}^{\sigma=\pi}=0\qquad\forall\sigma'. \tag{37}$$

From this equation we may conclude that both $\phi_0(\sigma)$ and $G^{-1}(\sigma,\sigma')$ are solutions but only for $\phi_0(+\pi)=\phi_0(-\pi)\neq0$; otherwise, the correction E_{corr} in (34) is not finite. This is a strong-coupling phase and therefore $\phi_0(\sigma)\neq0$ on the whole interval. Now we shall show that it is possible to construct an additional solution for ϕ_0 and G. We shall start from another functional identity (taken in the sense of principal part) [8]:

$$\frac{1}{f(\sigma)-f(\sigma')}\frac{1}{f(\sigma)-f(\sigma'')}+\frac{1}{f(\sigma')-f(\sigma)}\frac{1}{f(\sigma')-f(\sigma'')}$$
$$+\frac{1}{f(\sigma'')-f(\sigma)}\frac{1}{f(\sigma'')-f(\sigma')}=\pi^2(f(\sigma)-f(\sigma'))\delta(f(\sigma)-f(\sigma'')) \tag{38}$$

where $f(\sigma)$ again satisfies $-\pi\leqslant f(\sigma)\leqslant\pi$. Further treatment is close to the one-matrix problem [11]. Multiplying and integrating (38), we obtain

$$\begin{aligned}\fint d\sigma\,\phi_0(\sigma)(f'(\sigma))^2\frac{1}{f(\sigma)-f(\sigma')}\frac{1}{f(\sigma)-f(\sigma'')}&=\pi^2\phi_0(\sigma')\delta(\sigma-\sigma')-\frac{1}{f(\sigma')-f(\sigma'')}\\&\quad\times\left(\fint\frac{\phi_0(\sigma)(f'(\sigma))^2\,d\sigma}{f(\sigma')-f(\sigma)}-\fint\frac{\phi_0(\sigma)(f'(\sigma))^2\,d\sigma}{f(\sigma'')-f(\sigma)}\right).\end{aligned}\tag{39}$$

The condition for G^{-1} in (23b) can be satisfied if

$$\fint\frac{\phi_0(\sigma)(f'(\sigma))^2\,d\sigma}{f(\sigma)-f(\sigma')}=c_1+c_2f(\sigma')\tag{40}$$

c_1 and c_2 being constants. Using the properties of the Hilbert transform [10], we find that

$$f(\sigma)=-f(\sigma_0)\cos\frac{\pi}{T}t(\sigma)\qquad\sigma_0\leqslant\pi,\ \phi_0(\sigma)=0\qquad\text{if }|\sigma|>\sigma_0\tag{41}$$

where

$$t(\sigma)=\int_{-\sigma_0}^{\sigma}\frac{d\sigma}{\phi_0(\sigma)}\qquad\text{and}\qquad T=t(\sigma_0).\tag{42}$$

Then G^{-1} is given by

$$G^{-1}(\sigma,\sigma')=-2\ln\left|\cos\frac{\pi t(\sigma)}{T}-\cos\frac{\pi t(\sigma')}{T}\right|.\tag{43}$$

Inserting G^{-1} from (43) into (26), we obtain for the ground-state energy

$$\begin{aligned}E=\frac{\pi^2\lambda N^2}{3a}\int\phi_0^3(\sigma)\,d\sigma+\frac{N^2}{\lambda a}\int\phi_0(\sigma)S(\sigma)\,d\sigma-\frac{\lambda N^2}{12a}\\-\frac{\lambda}{a}\int d\sigma\,\phi_0(\sigma)\partial_\sigma\partial_{\sigma'}\ln\left|\frac{\cos\pi t(\sigma)/T-\cos\pi t(\sigma')/T}{\sin(\sigma-\sigma')/2}\right|_{\sigma=\sigma'}\end{aligned}\tag{44}$$

and hence we have

$$E_{\text{corr}}=\left(-\frac{\lambda\pi}{4Ta}\cot\frac{\pi t(\sigma)}{T}+\frac{\lambda}{4a}\partial_0\phi_0(\sigma)\right)\Bigg|_{\sigma=-\sigma_0}^{\sigma=\sigma_0}-\frac{\lambda\pi^2}{12Ta}+\frac{\lambda}{12a}-\frac{\lambda}{24a}\int_{-\sigma_0}^{\sigma_0}\frac{\partial_\sigma^2\phi_0^2}{\phi_0}\,d\sigma.\tag{45}$$

The mean value of the collective field $\phi_0(\sigma)$ is determined from (23a) as

$$\frac{\pi^2\lambda N^2}{a}\phi_0^2(\sigma)+\frac{N^2}{\lambda a}S(\sigma)-\frac{\lambda}{a}\partial_\sigma\partial_{\sigma'}\ln\left|\frac{\cos\pi t(\sigma)/T-\cos\pi t(\sigma')/T}{\sin(\sigma-\sigma')/2}\right|_{\sigma=\sigma'}=\mu_1.\tag{46}$$

Using the same $1/N$ arguments as before, we obtain the leading approximate solution

$$\tilde{\phi}_0(\sigma)=\frac{1}{\pi\lambda}\sqrt{\mu-S(\sigma)}.\tag{47}$$

From (23b) for G and G^{-1}, we obtain the condition

$$\phi_0(\sigma)\partial_\sigma G^{-1}(\sigma,\sigma')\big|_{\sigma=-\sigma_0}^{+\sigma_0}=0\qquad\forall\sigma'.\tag{48}$$

From this condition we may conclude that both $\phi_0(\sigma)$ and $G^{-1}(\sigma, \sigma')$ are solutions but only for $\phi_0(+\sigma_0) = \phi_0(-\sigma_0) = 0$. This is a weak-coupling phase.

3. The wavefunctional

We have assumed the Gaussian form of the wavefunctional. We shall show how to obtain its detailed form. The Jacobian of the transformation to collective variables is defined by the Hermiticity condition [3]

$$\frac{\delta \ln J}{\delta W_n} = -\sum_{n'} \Omega^{-1}(n, n')\omega(n'). \tag{49}$$

In terms of the collective field

$$\ln J = \frac{N^2}{2} \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \phi(\sigma) \ln\left(\sin^2 \frac{\sigma - \sigma'}{2}\right) \phi(\sigma') \tag{50}$$

the Schrödinger wavefunctional is given as

$$\Psi[\phi] = \mathrm{e}^{-(1/2)\ln J} \Phi[\phi] \tag{51}$$

where $\Phi[\phi]$ is the Gaussian ansatz (18).

Collecting the above equations, we obtain the Schrödinger wavefunctional (51) as

$$\Psi[\phi] = \exp\left[-\frac{N^2}{2} \int\int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \phi_0(\sigma) G^{-1}(\sigma, \sigma')\phi(\sigma') \right.$$
$$\left. -\frac{N^2}{4} \int\int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \phi(\sigma)\left(\ln \sin^2 \frac{\sigma - \sigma'}{2} + g^{-1}(\sigma, \sigma')\right)\Phi(\sigma')\right]. \tag{52}$$

It is easy to see that this wavefunctional is not singular for $\sigma = \sigma'$ owing to the Jacobian (50) which cancels the divergences. This is a unified treatment in the sense of its validity for both phases.

4. Examples

To show the efficiency of the collective-field method, we shall treat both the Jurkiewicz–Zalewski and the Manton actions.

(i) The action $S_{JZ}(\sigma) = 2\tan^2 \sigma/2$. Owing to this action, the model has only one phase, the weak one. The corresponding equation for $\phi_0(\sigma)$ from (46) is

$$\pi^2 g^2 \phi_0^2 + \frac{2}{g^2}\tan^2\frac{\sigma}{2} - \frac{1}{6}g^2 + \frac{g^2}{12}\left(\left(\frac{\partial_\sigma \phi_0}{\phi_0}\right)^2 + 2\partial_\sigma\left(\frac{\partial_\sigma \phi_0}{\phi_0}\right)\right)$$
$$-\frac{g^2\pi^2}{12T\phi_0^2} + \frac{g^2\pi^2}{4\pi^2\phi_0^2}\,\frac{1}{\sin^2 \pi t(\sigma)/T} = \mu. \tag{53}$$

This equation can be exactly solved as

$$\phi_0(\sigma) = \frac{N}{\pi\sqrt{2\lambda}}\left(2A + \frac{\lambda}{2} - \frac{2a^2}{\lambda}\tan^2\frac{\sigma}{2}\right)^{1/2} \tag{54}$$

where $A=[2+(\lambda^2/16N^2)]^{1/2}$, $\lambda=g^2N$ and the corresponding value for μ is

$$\mu=\frac{\lambda}{N}\left(\frac{N^2}{4}-\frac{1}{8}\right)+NA.$$

The corresponding ground-state energy is

$$\begin{aligned} E &= E_0+E_{\rm corr} \\ E_0 &= \frac{N^2}{4}A+\frac{N^2}{2A} \\ E_{\rm corr} &= \frac{\lambda}{8}+\frac{\lambda^2}{64A}. \end{aligned} \tag{55}$$

This is also the exact result for finite N, which can be shown by summing 'fermionic' orbitals using the BIPZ [4] method.

(*ii*) The Manton action $S(\sigma)=\sigma^2$. There are two phases [12]. *In the weak phase* $\lambda\leqslant\lambda_c=\pi^2/\sqrt{8}$, the exact solution of (46) is

$$\phi_0^{\rm w}(\sigma)=\left(\sqrt{2}N-\frac{1}{2g^2}\sigma^2\right)^{1/2} \tag{56}$$

$$E_0^{\rm w}=\frac{N^2}{\sqrt{2}}-\frac{\lambda N^2}{12}$$

$$E_{\rm corr}^{\rm w}=-\frac{g^2\pi^2}{12T}+\frac{g^2N}{12}-\frac{g^2}{24}\int\frac{\partial_\sigma(\phi_0^{\rm w})^2}{\phi_0^{\rm w}}\,{\rm d}\sigma \tag{57a}$$

$$=\frac{\lambda}{12}. \tag{57b}$$

Summation of the orbitals shows that this result is in complete agreement with the WKB treatment.

In the strong-coupling phase $\lambda\geqslant\pi^2/\sqrt{8}$, the approximate solution of (46) is

$$\tilde{\phi}_0^{\rm s}(\sigma)=\frac{1}{\pi g}\left(\mu+\frac{g^2}{4}-\frac{\sigma^2}{2g^2}\right)^{1/2} \tag{58}$$

and

$$E_0^{\rm s}=N^2\left(-\frac{\pi^3}{6\sqrt{2}\lambda^2}(A^2-1)^{3/2}+\frac{A^2\pi^2}{4}-\frac{\lambda}{12}\right).$$

$E_{\rm corr}$ is given by

$$E_{\rm corr}=-\frac{1}{2\sqrt{2}\pi(A^2-1)^{1/2}}+\frac{T}{24\pi^2\lambda}-\frac{\pi^2\lambda}{3T}+\frac{\lambda}{12} \tag{59}$$

where $T=2\sqrt{2}\pi\sin^{-1}1/A$ and A is defined by a transcendental equation arising from the normalization condition (24):

$$(A^2-1)^{1/2}+A^2\sin^{-1}\frac{1}{A}=\lambda\frac{\sqrt{2}}{\pi}. \tag{60}$$

The correction is finite except for $\lambda=\lambda_c$, i.e. $A=1$. It will be very interesting to investigate how to obtain this correction from the BIPZ method [4], taking into account complications arising from the boundary condition on wavefunctions at $\sigma=\pm\pi$.

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